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Invariance of the Cross Ratio Applied to Microwave Network Analysis

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Invariance of the Cross Ratio Applied to Microwave Network Analysis

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INVARIANCE OF THE CROSS RATIO APPLIED TO MICROWAVE NETWORK ANALYSIS

by

R. W. Beatty

ABSTRACT

The historical background and theory are given of the application of the mathematical principle "invariance of the cross-ratio" to microwave network analysis. Further developments to improve the accuracy of automatic network analyzers are suggested.

Key words: Admittance, anharmonic ratio; automatic network analyzers; cross ratio; impedance; microwave network analysis; reflection coefficient; scattering coefficient.

1. INTRODUCTION

It is well-known that the cross-ratio (or anharmonic ratio) of four distinct points in a complex plane is unchanged when these points are bilinearly transformed to four other points [1]. This principle has found use in a number of microwave measurement problems such as the determination of efficiencies of bolometer mounts [2], the determination of scattering coefficients of 2-ports [3], and in the measurement of reflection coefficients using computer controlled automatic systems [4].

It is anticipated that this technique will be further explored and exploited, especially with regard to the analysis and evaluation of errors in automatic network analyzers.[1] The purpose of this paper is to present a historical background and foundation for these anticipated developments.

1. Such work has already begun [4], but more will surely follow.

2. THE PRINCIPLE

The principle of invariance of the cross ratio is briefly illustrated as follows.

Consider the linear fractional transformation

$$z = \frac{aw + b}{cw + d}, \tag{1}$$

where z and w are complex variables, and a, b, c, and d are complex coefficients.

As shown in figure 1, four points in the w-plane are transformed by eq. (1) into four other points in the z-plane.

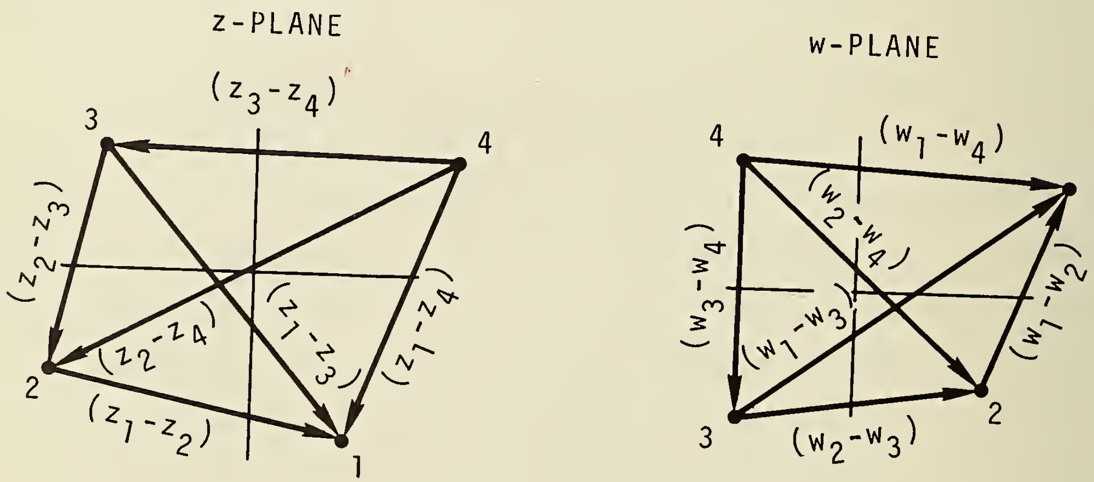


Figure 1. Four points in the w-plane transformed into four other points in the z-plane.

According to the principle of invariance, the cross-ratio of the w's is equal to the cross-ratio of the zs, or

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} . \qquad (2)$$

The validity of eq (2) has been demonstrated for example in [1].

3. DETERMINATION OF BOLOMETER MOUNT EFFICIENCY

Kerns [2] represented a bolometer mount as a 2-port as shown in figure 2.

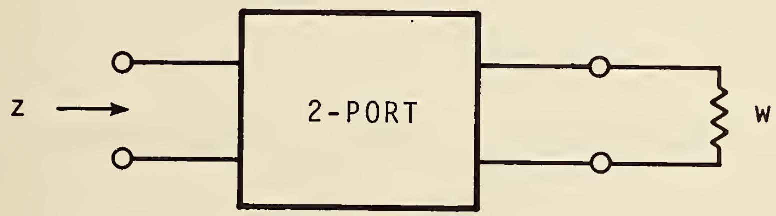


Figure 2. Representation of a bolometer mount by a 2-port.

He expressed the cross-ratio equivalence as follows

$$\frac{(z - z_3)(z_2 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(w - w_3)(w_2 - w_1)}{(w - w_1)(w_3 - w_2)} . \qquad (3)$$

Kerns solved for the coefficients a, b, c, and d in eq (1) and then determined the efficiency η_0 for the condition $w = w_0$.

He obtained

$$
\begin{aligned}
a &= \alpha(z_3 - rz_1)\\
b &= \alpha(rz_1 w_3 - z_3 w_1)\\
c &= \alpha(1 - r)\\
d &= \alpha(rw_3 - w_1)\,,
\end{aligned}
\tag{4}
$$

where $\alpha = [r(w_3 - w_1)(z_3 - z_1)]^{-\frac{1}{2}}$, $\qquad (5)$

$$
r = \frac{(w_2 - w_1)(z_3 - z_2)}{(z_2 - z_1)(w_3 - w_2)}\,,
\tag{6}
$$

$$
\text{and } ad - bc = 1.
\tag{7}
$$

Equations (5) and (7) are valid when reciprocity holds for the 2-port.

Further, the efficiency η_0 when $w = w_0$ is

$$
\eta_0 = \frac{\cos\phi}{N\cos(\theta + \phi)}\,,
\tag{8}
$$

where $$w_0 = |w_0|\, e^{j\theta}\,,$$

and $$N\, e^{j\phi} = \left(a + \frac{b}{w_0}\right)(cw_0 + d)^*\,,$$

where * denotes the complex conjugate.

In the above analysis it was assumed that z_1, z_2, and z_3 were measured corresponding to known values of w_1, w_2, and w_3 respectively. One could then determine a, b, c, and d, assuming reciprocity. Finally, a value of w_0 of load impedance was chosen for which a corresponding efficiency η_0 was calculated.

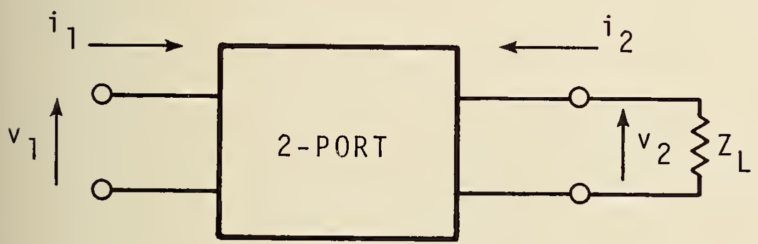
The above results could also have been obtained without using the principle of the invariance of the cross ratio. One could have instead solved three simultaneous equations similar in form to eq (1).

4. DETERMINING IMPEDANCE MATRIX OF A 2-PORT

The impedance matrix elements of a 2-port are defined by

$$\left\{\begin{aligned} v_1 &= z_{11}\, i_1 + z_{12}\, i_2 \\ v_2 &= z_{21}\, i_1 + z_{22}\, i_2 \ , \end{aligned}\right. \tag{9}$$

where the z's are impedances, and the v's and i's are voltage and currents in accordance with the conventions shown in figure 3.



$$v_2 = -\, i_2\, z_L$$

$$-i_2\, z_L = z_{21}\, i_1 + z_{22}\, i_2$$

Figure 3. Conventional directions for voltages and currents associated with a 2-port.

We can write

$$z_1 = \frac{v_1}{i_1} = z_{11} - \frac{z_{12}\, z_{21}}{z_{22} + z_L} = \frac{z_{11} z_L + (z_{11}\, z_{22} - z_{12}\, z_{21})}{z_L + z_{22}} \ . \tag{10}$$

This is of the same form as eq. (1), where $w = z_2$, $a = -z_{11}$, $b = \det z$, $c = 1$, and $d = z_{22}$. If we determine a, b, c, and d by the procedure used by Kerns, then

$$z_{11} = a, \quad z_{22} = d, \text{ and } z_{12} z_{21} = -(ad + bc). \tag{11}$$

Alternately, one could solve three simultaneous equations, each similar to eq (10), and obtain the elements of the impedance matrix.

5. DETERMINING SCATTERING COEFFICIENTS OF A 2-PORT

The elements of the scattering matrix for a 2-port are defined by

$$\begin{cases} b_1 = s_{11} a_1 + s_{12} a_2 \\ b_2 = s_{21} a_1 + s_{22} a_2 , \end{cases} \tag{12}$$

where the s's are scattering coefficients, and the a's and b's are respectively the incident and emergent voltage wave amplitudes in accordance with the conventions shown in figure 4.

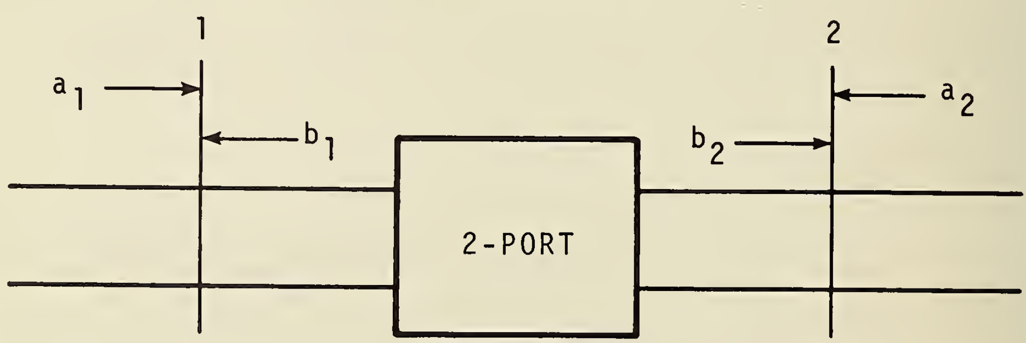


Figure 4. Conventional directions for incident and emergent voltage waves associated with a 2-port.

The scattering coefficients could be determined given the elements of the impedance matrix from published [5] conversion formulas. However, it is usually more convenient to measure the input reflection coefficients corresponding to three given terminating reflection coefficients. One will obtain three simultaneous equations of the following form

$$\Gamma_1 = \frac{r_{11}\,\Gamma_L + r_{12}}{r_{21}\,\Gamma_L + r_{22}} \tag{13}$$

where the Γ's denote the terminating reflection coefficients and the r's denote wave cascading coefficients [5]. The scattering coefficients are related to the wave cascading coefficients as follows

$$s = \frac{1}{r_{22}} \begin{bmatrix} r_{12} & \det r \\ & \\ 1 & -\,r_{21} \end{bmatrix} \tag{14}$$

Since eq (13) is of the same form as eq (1), one could follow Kerns' procedure to obtain the r-coefficients, then solve for the s's. Alternately, one could solve three simultaneous equations of the form

$$\Gamma_1 = s_{11} + \frac{s_{12}\,s_{21}\,\Gamma_L}{1 - s_{22}\,\Gamma_L} \tag{15}$$

to obtain the s' s. This was done [3] and the results were

$$s_{11} = \frac{\Gamma_{L1}\Gamma_{L2}\Gamma_{L3} (\Gamma_1 - \Gamma_2) + \Gamma_{L2}\Gamma_{L3}\Gamma_1 (\Gamma_2 - \Gamma_3) + \Gamma_{L3}\Gamma_{L1}\Gamma_2 (\Gamma_3 - \Gamma_1)}{\Gamma_{L1}\Gamma_{L2} (\Gamma_1 - \Gamma_2) + \Gamma_{L2}\Gamma_{L3} (\Gamma_2 - \Gamma_3) + \Gamma_{L3}\Gamma_{L1} (\Gamma_3 - \Gamma_1)} , \tag{16}$$

$$s_{22} = \frac{\Gamma_{L1} (\Gamma_2 - \Gamma_3) + \Gamma_{L2} (\Gamma_3 - \Gamma_1) + \Gamma_{L3} (\Gamma_1 - \Gamma_2)}{\Gamma_{L1}\Gamma_{L2} (\Gamma_1 - \Gamma_2) + \Gamma_{L2}\Gamma_{L3} (\Gamma_2 - \Gamma_3) + \Gamma_{L3}\Gamma_{L1} (\Gamma_3 - \Gamma_1)} , \tag{17}$$

and

$$s_{12}\, s_{21} = - \frac{(\Gamma_1 - \Gamma_2)(\Gamma_2 - \Gamma_3)(\Gamma_3 - \Gamma_1)(\Gamma_{L1} - \Gamma_{L2})(\Gamma_{L2} - \Gamma_{L3})(\Gamma_{L3} - \Gamma_{L1})}{[\Gamma_{L1}\Gamma_{L2} (\Gamma_1 - \Gamma_2) + \Gamma_{L2}\Gamma_{L3} (\Gamma_2 - \Gamma_3) + \Gamma_{L3}\Gamma_{L1} (\Gamma_3 - \Gamma_1)]^2} . \tag{18}$$

All of the results obtained so far could have been obtained without using the principle of the invariance of the cross ratio. It has been employed for convenience. In the following, it is even more convenient because four or more simultaneous equations must otherwise be solved.

6. DETERMINATION OF VOLTAGE REFLECTION COEFFICIENT

Consider a linear circuit representing a measuring instrument and an object under test whose voltage reflection coefficient is to be measured. Such a circuit is shown in figure 5.

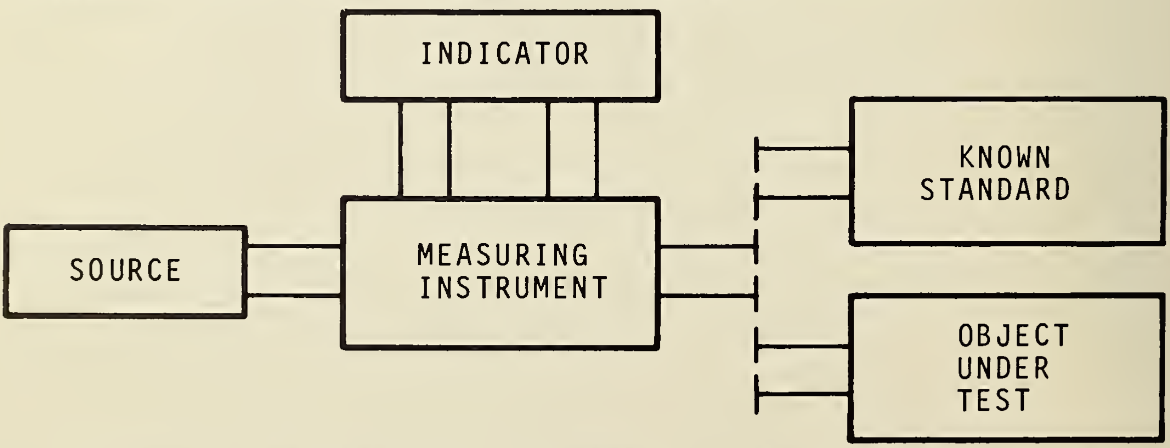


Figure 5. Representation of measuring instrument and objects to be measured.

Suppose that the response of the measuring instrument as observed at the indicator is represented by V, a complex number. It is linearly related to the reflection coefficient Γ of the object to be measured by the equation[2]

$$V = \frac{A\,\Gamma + B}{C\,\Gamma + d} , \tag{19}$$

where A, B, C, and D, are coefficients whose values are independent of V and Γ.

It follows that the cross-ratio of the V's, $\{V\}$ equals the cross-ratio of the Γ's, $\{\Gamma\}$. Thus

$$\frac{(V_1 - V_2)(V_3 - V_4)}{(V_2 - V_3)(V_4 - V_1)} = \frac{(\Gamma_1 - \Gamma_2)(\Gamma_3 - \Gamma_4)}{(\Gamma_2 - \Gamma_3)(\Gamma_4 - \Gamma_1)}, \text{ or } \{V\} = \{\Gamma\}. \tag{20}$$

If Γ_1, Γ_2, and Γ_3 correspond to three known standards, and we measure V_1, V_2, V_3, and V_4, then we can solve for Γ_4 as follows.

Let $$r = \frac{(V_1 - V_2)(V_3 - V_4)(\Gamma_2 - \Gamma_3)}{(V_2 - V_3)(V_4 - V_1)(\Gamma_1 - \Gamma_2)} = \frac{\Gamma_3 - \Gamma_4}{\Gamma_4 - \Gamma_1} . \tag{21}$$

Then $$\Gamma_4 = \frac{\Gamma_3}{1 + r} \cdot \left(1 + r\,\frac{\Gamma_1}{\Gamma_3}\right). \tag{22}$$

2. This is generally true for any linear measuring instrument for any type of voltage response. For example, V may be the complex ratio of the voltage wave outputs of two directional couplers, one coupling to the forward going wave, and the other to the wave reflected from the load [6].

The three known standards may be chosen in a number of ways, depending upon what is convenient and available. Some of the choices that might be considered are shown in Table I. In the table, sequence A1 corresponds to the general case described above, where the voltage reflection coefficients of the three known standards are designated as Γ_{s1}, Γ_{s2}, and Γ_{s3}.

TABLE I. VARIOUS SEQUENCES FOR CONNECTING THREE STANDARDS THEN THE UNKNOWN

ORDER OF CONNECTION	SEQUENCE				
	A 1	B 1	C 1	D 1	E 1
1	Γ_{s1}	$e^{j\theta_1}$	- 1	- 1	1
2	Γ_{s2}	$e^{j\theta_2}$	1	1	0
3	Γ_{s3}	$e^{j\theta_3}$	Γ_s	0	Γ_s
4	Γ_U	Γ_U	Γ_U	Γ_U	Γ_U

The solutions for Γ_U corresponding to the sequences in Table I are as follows.

$$\Gamma_U = \frac{r_{A1}\Gamma_{s1} + \Gamma_{s3}}{r_{A1} + 1} \tag{23}$$

where
$$r_{A1} = \{V\}_{A1} \frac{\Gamma_{s2} - \Gamma_{s3}}{\Gamma_{s1} - \Gamma_{s2}} = \frac{\Gamma_{s3} - \Gamma_U}{\Gamma_U - \Gamma_{s1}}$$

and
$$\{V\}_{A1} = \left\{ \frac{(V_1 - V_2)(V_3 - V_U)}{(V_2 - V_3)(V_U - V_1)} \right\}$$

$$\Gamma_U = \frac{r_{B1} e^{j\theta_1} + e^{j\theta_3}}{r_{B1} + 1} , \tag{24}$$

where $$r_{B1} = \{V\}_{B1} \frac{e^{j\theta_2} - e^{j\theta_3}}{e^{j\theta_1} - e^{j\theta_2}} .$$

Note that only one standard, a sliding short-circuit is required for the above technique using eq. (24) and the B_1 sequence of table I. Its positions in terms of the guide wavelength must be accurately determined. It is convenient to choose equi-spaced positions so that $r_{B1} = \{V\}_{B1}$.

$$\Gamma_U = \frac{\Gamma_s - r_{c1}}{1 + r_{c1}} \tag{25}$$

where $$r_{c1} = \{V\}_{c3} \cdot \frac{1 - \Gamma_s}{(-2)} .$$

$$\Gamma_U = \frac{-r_{D1}}{1 + r_{D1}} \tag{26}$$

where $$r_{D1} = -\tfrac{1}{2} \{V\}_{D1}$$

$$\Gamma_U = \frac{r_{E1} + \Gamma_s}{r_{E1} + 1} , \tag{27}$$

where $$r_{E1} = -\{V\}_{E1} \cdot \Gamma_s .$$

Other formulas may be obtained using the cross ratios shown in the Appendix.

In sequence B1, the first three conditions could be closely realized by sliding a short-circuit to three positions inside a waveguide. In sequences C1 and D1, the first two conditions might be closely realized by connecting first a flat plate short-circuit, then a quarter-wavelength short-circuited section of waveguide. In sequences D1 and E1, a non-reflecting termination ($\Gamma = 0$) can be closely realized by means of an adjustable sliding termination [7]. Note that the first two conditions in sequences C1 and D1 might also have been closely realized by using first a flat plate short-circuit, then inserting a quarter-wavelength section of waveguide.

The sequences in Table II are based upon the following technique. If one inserts a quarter-wavelength section of waveguide between the unknown and the output port of the measuring instrument, we obtain the reflection coefficient $-\Gamma_u$. Then it is necessary to connect two additional known standards in order to determine a cross-ratio. (We could alternately connect one standard directly, then insert a quarter-wavelength section of waveguide.)

Table II shows a number of sequences which might be chosen, using this quarter-wave technique.

TABLE II. VARIOUS SEQUENCES FOR CONNECTING ONE OR TWO STANDARDS THEN THE UNKNOWN USING A QUARTER-WAVE TECHNIQUE.

ORDER OF CONNECTION	SEQUENCE					
	M 1	N 1	O 1	P 1	Q 1	R 1
1	Γ_{s1}	$-\Gamma_s$	-1	1	$-\Gamma_s$	Γ_s
2	Γ_{s2}	Γ_s	1	0	Γ_s	0
3	$-\Gamma_U$	$-\Gamma_U$	$-\Gamma_U$	$-\Gamma_U$	0	$-\Gamma_U$
4	Γ_U	Γ_U	Γ_U	Γ_U	Γ_U	Γ_U

As Townsend [1] has noted there are 24 possible variations of each sequence shown, of which only 6 give different cross-ratios. It is further noted in the Appendix that 3 of the 6 are reciprocals of the other 3. Thus we need consider only 3 variations M_1, M_2, and M_3, of the M sequence, for example. The cross-ratios of the V's are given in the Appendix for 3 variations of each sequence. It turns out that the cross-ratio of the 3rd sequence in each case is the most interesting from our point of view.

The solutions for Γ_U corresponding to the sequences M3, N3, O3, P3, Q3, and R3 (See Appendix) are as follows[3]:

$$\Gamma_U = \left(\frac{\{V\}_{M3}+1}{\{V\}_{M3}-1} \cdot \frac{\Gamma_{s1}-\Gamma_{s2}}{2}\right) \pm \sqrt{\left(\frac{\{V\}_{M3}+1}{\{V\}_{M3}-1} \cdot \frac{\Gamma_{s1}-\Gamma_{s2}}{2}\right)^2 + \Gamma_{s1}\Gamma_{s2}}$$

$$\Gamma_U = \frac{1+\sqrt{\{V\}_{N3}}}{1-\sqrt{\{V\}_{N3}}} \cdot \Gamma_s, \text{ or } = \frac{1-\sqrt{\{V\}_{N3}}}{1+\sqrt{\{V\}_{N3}}} \cdot \Gamma_s . \qquad (29)$$

It is worth noting that if $\Gamma_s \to \Gamma_U$, then $\{V\}_{N3} \to 0$, as can be seen from eq. (24A), and errors in measuring $\{V\}_{N3}$ produce small errors in determining Γ_U using eq. (29).

$$\Gamma_U = \frac{1+\sqrt{\{V\}_{O3}}}{1-\sqrt{\{V\}_{O3}}}, \text{ or } = \frac{1-\sqrt{\{V\}_{O3}}}{1+\sqrt{\{V\}_{O3}}} . \qquad (30)$$

$$\Gamma_U = \frac{\{V\}_{P3}+1}{\{V\}_{P3}-1} . \qquad (31)$$

$$\Gamma_U = \frac{1-\{V\}_{Q3}}{1+\{V\}_{Q3}} \cdot \Gamma_s . \qquad (32)$$

$$\Gamma_U = \frac{1+\{V\}_{R3}}{1-\{V\}_{R3}} \cdot \Gamma_s . \qquad (33)$$

3. As can be deduced from the A3 sequence in the Appendix, one obtains the M3 sequence by interchanging the order of Γ_{s2} and $-\Gamma_U$ in the M1 sequence.

In one application [4] the reflection coefficient Γ_T of a sliding termination was first measured using an "O"-sequence. (Note that Γ_U in "O"-sequence corresponds to Γ_T). Then the sliding termination was fixed and used as a known standard having a reflection coefficient Γ_s in a "C"-sequence to determine the Γ_U of an unknown. The sliding loads and sliding short-circuits used in this technique are either commercially available or easily fabricated.

So far, it has been taken for granted that both magnitudes and phases of the V's and Γ's are determined. However, it is possible to extract only the magnitudes $|z_U|$, $|y_U|$, and $|\Gamma_U|$ with more limited information.

If we can measure only the magnitudes of ratios such as

$$\left| \frac{V_1 - V_2}{V_2 - V_3} \right|,$$

using for example a two-channel nulling system [8], we can then determine the magnitude of a cross-ratio $\{V\}$. If we use the quarter-wavelength technique described by Little and Ellerbruch [8], we will obtain V's corresponding to Γ's of -1, 1, $-\Gamma_U$, and Γ_U. We can then measure $|\{V\}_0|$. As shown in the appendix, this will give us[4]

$$|1 - y_U|^2, \ |1 - z_U|^2, \ |y_U|^2, \ \left|\frac{1}{1 - z_U}\right|^2, \ |z_U|^2, \text{ or } \left|\frac{1}{1 - y_U}\right|^2 \text{ directly,}$$

depending upon what order we use in connections. The determination of $|\Gamma_U|$ by this technique is also possible but involves more data taking and calculation. If we denote $\{V\}_{o2}$ to correspond to one order of Γ's $[-1, 1, \Gamma_U$, and $-\Gamma_U]$, and we denote $\{V_{o7}\}$ to correspond to another order of Γ's $[-1, \Gamma_U, 1, -\Gamma_U]$, then we can write

4. Here, and in the Appendix, z_U and y_U denote, respectively, normalized impedance and admittance, such that $\Gamma_U = \dfrac{z_U - 1}{z_U + 1} = \dfrac{1 - y_U}{1 + y_U}$.

$$|\Gamma_U|^2 = \frac{1 - \sqrt{1 - \left(\dfrac{|\{V\}_{02}|}{1 + |\{V\}_{07}|}\right)^2}}{1 + \sqrt{1 + \left(\dfrac{|\{V\}_{02}|}{1 + |\{V\}_{07}|}\right)^2}} \tag{34}$$

This is more complicated than the procedure described by Little and Ellerbruch [8], but it requires no approximations and holds for a non-ideal system.

An alternate technique is indicated following eq (30A) in the appendix.

7. TOPICS FOR FURTHER RESEARCH

There are immediate steps which can be taken to improve the accuracy of measurements with automatic network analyzers, and future steps which will require more research.

Present calibration procedures employing short circuits assume losslessness. A small improvement in accuracy can be immediately realized by taking losses into account. One can employ quarter-wavelength short-circuits [9] and can also correct for losses in quarter-wavelength sections of waveguide.

It appears likely that greater improvements in accuracy might be realized by choosing different sequences than have been employed. One would guess that sequences employing standards in which Γ_s was not greatly different than Γ_U might have lower errors. This was noted in connection with eq. (29), for example.

Thus, further research is needed to determine how given errors in measuring the V's and in determining the Γ's of the known standards propagate in using different sequences. More research is also needed to develop suitable standards having $|\Gamma|$'s intermediate between zero and unity.

8. ACKNOWLEDGEMENT

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9. APPENDIX

In the following, three forms for the cross ratio $\{V\}$ are given for each sequence listed in Tables I and II. All of the forms listed can be considered as special cases of $\{V\}_A$.

The order in which the sequences are taken are listed below

$\{V\}_{A1}$	$\{V\}_{A2}$	$\{V\}_{A3}$	$\{V\}_{A4}$	$\{V\}_{A5}$	$\{V\}_{A6}$
Γ_{s1}	Γ_{s1}	Γ_{s1}	Γ_{s1}	Γ_{s1}	Γ_{s1}
Γ_{s2}	Γ_{s2}	Γ_{s3}	Γ_U	Γ_{s3}	Γ_U
Γ_{s3}	Γ_U	Γ_{s2}	Γ_{s3}	Γ_U	Γ_{s2}
Γ_U	Γ_{s3}	Γ_U	Γ_{s2}	Γ_{s2}	Γ_{s3}

Note that

$$\{V\}_{A4} = \frac{1}{\{V\}_{A1}} \tag{1A}$$

$$\{V\}_{A5} = \frac{1}{\{V\}_{A2}} \tag{2A}$$

$$\{V\}_{A6} = \frac{1}{\{V\}_{A3}} \tag{3A}$$

In view of the reciprocal relationships above, only the first three forms are required.

$$\{V\}_{A1} = \frac{\Gamma_{s1} - \Gamma_{s2}}{\Gamma_{s2} - \Gamma_{s3}} \cdot \frac{\Gamma_{s3} - \Gamma_U}{\Gamma_U - \Gamma_{s1}} \tag{4A}$$

$$\{V\}_{A2} = \frac{\Gamma_{s1} - \Gamma_{s2}}{\Gamma_{s2} - \Gamma_U} \cdot \frac{\Gamma_U - \Gamma_{s3}}{\Gamma_{s3} - \Gamma_{s1}} \tag{5A}$$

$$\{V\}_{A3} = \frac{\Gamma_{s1} - \Gamma_{s3}}{\Gamma_{s3} - \Gamma_{s2}} \cdot \frac{\Gamma_{s2} - \Gamma_U}{\Gamma_U - \Gamma_{s1}} \tag{6A}$$

As noted above, the following forms are all special cases of $\{V\}_A$.

$$\{V\}_{B1} = \frac{e^{j\theta_1} - e^{j\theta_2}}{e^{j\theta_2} - e^{j\theta_3}} \cdot \frac{e^{j\theta_3} - \Gamma_U}{\Gamma_U - e^{j\theta_1}} \tag{7A}$$

$$\{V\}_{B2} = \frac{e^{j\theta_1} - e^{j\theta_2}}{e^{j\theta_2} - \Gamma_U} \cdot \frac{\Gamma_U - e^{j\theta_3}}{e^{j\theta_3} - e^{j\theta_1}} \tag{8A}$$

$$\{V\}_{B3} = \frac{e^{j\theta_1} - e^{j\theta_3}}{e^{j\theta_3} - e^{j\theta_2}} \cdot \frac{e^{j\theta_2} - \Gamma_U}{\Gamma_U - e^{j\theta_1}} \tag{9A}$$

The above expressions simplify if $\theta_3 - \theta_2 = \theta_2 - \theta_1$.

$$\{V\}_{C1} = \frac{-2}{1 - \Gamma_s} \cdot \frac{\Gamma_s - \Gamma_U}{1 + \Gamma_U} = \frac{z_U - z_s}{z_U} \tag{10A}$$

$$\{V\}_{C2} = \frac{-2}{1 - \Gamma_U} \cdot \frac{\Gamma_U - \Gamma_s}{1 + \Gamma_s} = \frac{z_s - z_U}{z_s} \tag{11A}$$

$$\{V\}_{C3} = \frac{1 + \Gamma_s}{1 - \Gamma_s} \cdot \frac{1 - \Gamma_U}{1 + \Gamma_U} = \frac{z_s}{z_U} \tag{12A}$$

The latter sequence appears attractive for measuring Z_U.

$$\{V\}_{D1} = \frac{2\Gamma_U}{1 + \Gamma_U} = 1 - y_U \tag{13A}$$

$$\{V\}_{D2} = \frac{-2\Gamma_U}{1 - \Gamma_U} = 1 - z_U \tag{14A}$$

$$\{V\}_{D3} = \frac{1 - \Gamma_U}{1 + \Gamma_U} = y_U \tag{15A}$$

The latter sequence and the one giving the reciprocal cross-ratio are of interest in determining y_U, z_U, $|y_U|$, and $|z_U|$.

$$\{V\}_{E1} = \frac{\Gamma_s - \Gamma_U}{\Gamma_s (1 - \Gamma_U)} = \frac{z_s - z_U}{z_s - 1} \tag{16A}$$

$$\{V\}_{E2} = \frac{\Gamma_U - \Gamma_s}{\Gamma_U (1 - \Gamma_s)} = \frac{z_U - z_s}{z_U - 1} \tag{17A}$$

$$\{V\}_{E3} = \frac{\Gamma_U (1 - \Gamma_s)}{\Gamma_s (1 - \Gamma_U)} = \frac{z_U - 1}{z_s - 1} \tag{18A}$$

$$\{V\}_{M1} = \frac{\Gamma_{s1} - \Gamma_{s2}}{\Gamma_{s2} + \Gamma_U} \cdot \frac{2\Gamma_U}{\Gamma_{s1} - \Gamma_U} \tag{19A}$$

$$\{V\}_{M2} = \frac{\Gamma_{s1} - \Gamma_{s2}}{\Gamma_U - \Gamma_{s2}} \cdot \frac{2\Gamma_U}{\Gamma_U + \Gamma_{s1}} \tag{20A}$$

$$\{V\}_{M3} = \frac{\Gamma_{s1} + \Gamma_U}{\Gamma_{s2} + \Gamma_U} \cdot \frac{\Gamma_{s2} - \Gamma_U}{\Gamma_{s1} - \Gamma_U} \tag{21A}$$

$$\{V\}_{N1} = \frac{4\Gamma_U \Gamma_s}{(\Gamma_U + \Gamma_s)^2} = \frac{(z_U{}^2 - 1)(z_s{}^2 - 1)}{(z_U z_s - 1)^2} \tag{22A}$$

$$\{V\}_{N2} = \frac{-4\Gamma_U \Gamma_s}{(\Gamma_U - \Gamma_s)^2} = \frac{(z_U^2 - 1)(z_s^2 - 1)}{(z_U - z_s)^2} \tag{23A}$$

$$\{V\}_{N3} = \left(\frac{\Gamma_U - \Gamma_s}{\Gamma_U + \Gamma_s}\right)^2 = \left(\frac{z_U - z_s}{z_U z_s - 1}\right)^2 \tag{24A}$$

$$\{V\}_{O1} = \frac{4\Gamma_U}{(1 + \Gamma_U)^2} = 1 - y_U^2 \tag{25A}$$

$$\{V\}_{O2} = \frac{-4\Gamma_U}{(1 - \Gamma_U)^2} = 1 - z_U^2 \tag{26A}$$

$$\{V\}_{O3} = \left(\frac{1 - \Gamma_U}{1 + \Gamma_U}\right)^2 = y_U^2 \tag{27A}$$

The latter sequence and the one giving the reciprocal cross-ratio are of interest in determining y_U, z_U, $|y_U|$, and $|z_U|$.

$$\{V\}_{P1} = \frac{2}{1 - \Gamma_U} = z_U + 1 \tag{28A}$$

$$\{V\}_{P2} = \frac{2}{1 + \Gamma_U} = y_U + 1 \tag{29A}$$

$$\{V\}_{P3} = -\frac{1 + \Gamma_U}{1 - \Gamma_U} = -z_U \tag{30A}$$

Note that we could obtain $|\Gamma_U|$ from the ratio of $|\{V\}_{O2}|$ and $|\{V\}_{P1}|$.

$$\{V\}_{Q1} = \frac{2\Gamma_U}{\Gamma_U + \Gamma_s} = \frac{(z_U - 1)(z_s + 1)}{z_U z_s - 1} \tag{31A}$$

$$\{V\}_{Q2} = \frac{2\Gamma_U}{\Gamma_U - \Gamma_s} = \frac{(z_U - 1)(z_s + 1)}{z_U - z_s} \tag{32A}$$

$$\{V\}_{Q3} = \frac{\Gamma_s - \Gamma_U}{\Gamma_s + \Gamma_U} = \frac{z_s - z_U}{z_s z_U - 1} \tag{33A}$$

$$\{V\}_{R1} = \frac{2\Gamma_s}{\Gamma_s - \Gamma_U} = \frac{(z_s - 1)(z_U + 1)}{z_s - z_U} \tag{34A}$$

$$\{V\}_{R2} = \frac{2\Gamma_s}{\Gamma_s + \Gamma_U} = \frac{(z_s - 1)(z_U + 1)}{z_U z_s - 1} \tag{35A}$$

$$\{V\}_{R3} = \frac{\Gamma_U + \Gamma_s}{\Gamma_U - \Gamma_s} = \frac{z_U z_s - 1}{z_U - z_s} \tag{36A}$$

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